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Geometric and Probabilistic Estimates for Entropy and Approximation Numbers of Operators

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It is an open problem whether the entropy numbers $e_n(T)$ of continuous linear operators $T: X \to Y$ are essentially self-dual, i.e., $e_n(T) \sim e_n(T^*)$. We give a positive result in the case that X and Y^* are of type 2, using volume estimates. This generalizes a result of Carl (On Gelfand, Kolmogorov, and entropy numbers of operators acting between special Banach spaces, University of Jena, Jena, East Germany, 1983, preprint). Moreover, we derive bounds for the approximation numbers $a_n(T)$ of T by probabilistic averaging. The formulas are applied to determine the exact asymptotic order of the approximation numbers of the formal identity map between various sequence spaces as well as tensor product spaces. In the special case of l_p^n, the result was first proved by Gluskin (*Mat. Sb.* **120** (1983), 180–189. [Russian]) using a different method.

0. INTRODUCTION

We recall some basic definitions and notions. If X is a Banach space, we denote its unit ball by B_X. The n-dimensional l_p-spaces will be denoted by l_p^n, $1 \leqslant p \leqslant \infty$, the conjugate index by $p' = p/(p-1)$. All operators $T: X \to Y$ will be continuous linear maps between real or complex Banach spaces. Given such $T: X \to Y$, we let

$$a_n(T) := \inf\{ \|T - T_n\| \mid T_n : X \to Y, \text{ rank } T_n < n\}$$

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stand for the *approximation numbers*,

$$c_n(T) := \inf\{ \|T|_{X_n}\| \mid X_n \subseteq X \text{ of codim } X_n < n \}$$

for the *Gelfand numbers*,

$$d_n(T) := \inf_{\substack{L \subseteq Y \\ \dim L < n}} \sup_{\|x\| = 1} \inf_{y \in L} \|Tx - y\|$$

for the *Kolmogorov numbers*, and

$$e_n(T) := \inf\left\{ \sigma > 0 \mid T(B_X) \subseteq \bigcup_{i=1}^{2^{n-1}} (\{y_i\} + \sigma B_Y) \qquad \text{for some } y_1, ..., y_{2^{n-1}} \in F \right\}$$

for the *entropy numbers* of T, $n \in N$. Any of these sequences $s_n \in \{a_n, c_n, d_n, e_n\}$ satisfies the relations

$$\|T\| = s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0,$$

$$s_{n+m-1}(S+T) \leqslant s_n(S) + s_m(T); \; S, T: X \to Y,$$

$$s_{n+m-1}(ST) \leqslant s_n(S)\, s_m(T); \; S: Y \to Z, \; T: X \to Y.$$

Moreover,

$$a_n(T) \geqslant c_n(T), \qquad a_n(T) \geqslant d_n(T),$$

with equality for Hilbert space operators T, in which case we denote these *singular numbers* by $s_n(T)$. For $1 \leqslant p \leqslant \infty$, let c_p^n denote the space of operators $L(l_2^n, l_2^n)$ normed by

$$\|T\|_{c_p^n} := \left(\sum_{j \in N} s_j(T)^p \right)^{1/p}.$$

In particular, $\|\cdot\|_{c_1^n}$ is the nuclear and $\|\cdot\|_{c_\infty^n}$ the operator norm. All these facts about s-numbers can be found in Pietsch [13]. If $i: Y \to Z$ is an injection, $q: W \to X$ a surjection , and $T: X \to Y$ continuous linear, the entropy numbers satisfy

$$e_n(Tq) = e_n(T), \qquad \tfrac{1}{2} e_n(T) \leqslant e_n(iT) \leqslant e_n(T);$$

they are *surjective* and (up to a factor of $\frac{1}{2}$), *injective*.

Given sequences $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$, we write

$$s_n \sim t_n$$

if there are constants $c, d > 0$ with

$$c\, s_n \leqslant t_n \leqslant d\, s_n, \qquad n \in \mathbb{N}.$$

Generally vol_n will denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$. By $r_n(t)$ we shall denote the nth Rademacher function and by $(g_n(\omega))_{n\in\mathbb{N}}$ a sequence of independent standard Gaussian variables on some probability space, i.e., each g_n has density function

$$(2\pi)^{-1/2}\, e^{-t^2/2}.$$

A Banach space is *of* (*Gaussian*) *type* 2 provided that there is $c>0$ such that for every finite sequence $(x_i)_{i=1}^n \subseteq X$

$$\left(\mathbb{E}\left\|\sum_{i=1}^n g_i x_i\right\|^2\right)^{1/2} \leqslant c\left(\sum_{i=1}^n \|x_i\|^2\right)^{1/2},$$

where $\mathbb{E}$ denotes the expectation. The best type 2 constant c will be denoted by $T_2(X)$. The space is of (*Rademacher*) *type* 2 if the same holds for Rademacher functions r_i instead of Gaussian functions g_i. Both notions coincide in type 2 spaces, since by Maurey and Pisier [12] there is an increasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\sqrt{2/\pi}\left(\mathbb{E}\left\|\sum_{i=1}^n r_i x_i\right\|^2\right)^{1/2} \leqslant \left(\mathbb{E}\left\|\sum_{i=1}^n g_i x_i\right\|^2\right)^{1/2} \leqslant f(T_2(X))\left(\mathbb{E}\left\|\sum_{i=1}^n r_i x_i\right\|^2\right)^{1/2}.$$

Let R_n be the natural projection of $L_2((0,1);X)$ onto $\overline{\text{span}}\{g_i(t)\,x_i \mid i=1,\ldots,n,\ x_i\in X\}$. X is called *K-convex* if $\gamma(X) := \sup_n \|R_n\| < \infty$. Any space X of type 2 is K-convex, and every quotient space of X, X/Y for $Y\subseteq X$, is again of type 2 with a uniform estimate

$$T_2(X/Y) \leqslant T_2(X)\,\gamma(X) \leqslant g(T_2(X))$$

for some increasing function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$. A detailed account on the K-convexity property may be found in Pisier [17]. For the notion of 2-convex lattices and 1-unconditional as well as 1-symmetric bases we refer to Lindenstrauss and Tzafriri [10].

If $\{x_i\}$ is a 1-unconditional basis of X, we let $\lambda_X(n) := \|\sum_{i=1}^n x_i\|$. $\{x_i^*\}$ will denote the biorthogonal sequence in the dual space X^*. For the notions of p-summing operators (Π_p, π_p) for $1\leqslant p\leqslant\infty$ and Hilbert-factorizable operators (Γ_2, γ_2) cf. [10 or 13].

The unit ball of a Banach space X is denoted by $B_X = \{x\colon \|x\|\leqslant 1\}$. If $\dim(X)=n$, then B_{X^*} will be identified with $\{y\in R^n\colon |(x,y)\leqslant 1 \text{ for all } x\in B_X\}$.

1. Entropy Numbers

It is an open problem whether the entropy numbers are essentially self-dual, i.e., whether asymptotically

$$e_n(T) \sim e_n(T^*), \qquad T \in L(X, Y) \tag{1.0}$$

holds for all continuous linear operators T. Direct calculations have shown (1.0) to be true for certain diagonal operators in sequence spaces. Carl [1] obtained a positive result for operators T between certain Banach spaces if $e_n(T) \sim n^{-\alpha}$ for some $\alpha > 0$. We generalize his result in

THEOREM 1.1. *Let X and Y be Banach spaces such that X and Y^* are of type 2. Then for any $T \in L(X, Y)$*

$$d^{-1} e_{[nc]}(T) \leqslant e_n(T^*) \leqslant d e_{[n/c]}(T), \tag{1.1}$$

where $c = \ln(18\, T_2(X)\, T_2(Y^*)\, \gamma(X)\, \gamma(Y^*))$ and $d = A(T_2(X)\, T_2(Y^*))^2$ only depend on X and Y, A being an absolute constant.

We need some lemmas. Volume estimates will yield a duality statement for $e_n(T)$ for rank n operators T. An approximation argument together with estimates of approximation numbers by entropy numbers will then settle the general case for arbitrary $T \in L(X, Y)$.

LEMMA 1.2. *Let X be a real n-dimensional space, B_2 be the ellipsoid of maximal volume contained in B_{X^*}. Then*

$$\left(\frac{\mathrm{vol}_n B_{X^*}}{\mathrm{vol}_n B_2}\right)^{1/n} \leqslant \left(\frac{\mathrm{vol}_n B_2}{\mathrm{vol}_n B_X}\right)^{1/n} \leqslant T_2(X).$$

This lemma follows basically from the argument in the proof of Proposition 2.1 of Gordon and Reisner [6]. The left volume ratio was also estimated by $T_2(X)$ by Tomczak and Jaegermann, cf. Rogalski [14]. For the sake of completeness we indicate how the proof in [6] is modified. It was recently proved by Bourgain and Milman that $(\mathrm{vol}_n B_X \,\mathrm{vol}_n B_{X^*})^{1/n} \sim (\mathrm{vol}_n B_2)^{2/n}$ (cf. [16]).

Proof. Let $E := X^*$ and B_2 be the ellipsoid of maximal volume in B_E which defines the Hilbert space l_2^n. We denote the formal identity map by v: $l_2^n \to E$ (John map). Hence $\|v\| \leqslant 1$, $\|v^{-1}\| \leqslant \sqrt{n}$. By Santalo's inequality [15]

$$\left(\frac{\mathrm{vol}_n B_E}{\mathrm{vol}_n B_2}\right)^{1/n} \leqslant \left(\frac{\mathrm{vol}_n B_2}{\mathrm{vol}_n B_{E^*}}\right)^{1/n}.$$

B_2 is also the ellipsoid of minimal volume containing $B_{E^*} = B_X$, and $v^{*-1}: l_2^n \to E^* = X$ the John map for X. The proof of Proposition 2.1 of [6, (2.9)] yields, using this fact about v^{*-1},

$$\left(\frac{\mathrm{vol}_n B_2}{\mathrm{vol}_n B_X}\right)^{1/n} \leqslant n^{-1/2} \left(\mathbb{E} \left\| \sum_1^n g_i v^{*-1}(e_i^*) \right\|^2\right)^{1/2},$$

which is estimated by $T_2(X)$ as in [6]. ▮

LEMMA 1.3. *Let X and Y be Banach spaces with X, Y^* being of type 2. Then for all operators $T \in L(X, Y)$ of rank $T \leqslant n$*

$$e_{[cn]}(T) \leqslant 4e_n(T^*), \tag{1.2}$$

where $c = \log_2(18\, T_2(X)\, T_2(Y^*)\, \gamma(X)\, \gamma(Y^*))$.

Proof. We will assume that X and Y are real Banach spaces. In the complex case, n-dimensional volume estimates have to be replaced by $(2n)$-dimensional real volume arguments.

(i) We first assume that X and Y are n-dimensional. Let H and K denote the n-dimensional Hilbert spaces the unit balls of which are the maximal volume ellipsoids contained in B_{X^*} and B_Y. Then B_H is also the minimal volume ellipsoid containing B_X. Let $i: X \to H$, $j: K \to Y$ denote the formal identity maps. Thus $\|i\| = \|j\| = 1$. The polar decomposition theorem shows that the entropy numbers of any $S: H \to K$ are the same as those of the diagonal operator in l_2^n induced by the singular numbers of S. The latter being self-dual, we have $e_n(S) = e_n(S^*)$. Considering $T: X \to Y$ as a map between Hilbert spaces, we find with $S = j^{-1} T i^{-1}$,

$$a := \log_2(3T_2(X)), \qquad b := \log_2(3T_2(Y^*)), \qquad m := [an] + [bn] + n + 2$$

$$\begin{aligned} e_m(T: X \to Y) &\leqslant e_m(j^{-1} T i^{-1}: H \to K) \\ &= e_m(i^{-1*} T^* j^{-1*}: K \to H) \\ &\leqslant e_{[an]+2}(i^{-1*}: X^* \to H) \\ &\quad \times e_{[bn]+2}(j^{-1*}: K \to Y^*)\, e_n(T^*: Y^* \to X^*). \end{aligned} \tag{1.3}$$

Choose in B_K a maximal system $M = \{z_i\}_{i=1}^L$ of points such that

$$\|z_i - z_j\|_{Y^*} > 1 \qquad \text{for all } 1 \leqslant i \neq j \leqslant L.$$

Then $\bigcup_{i=1}^L (\{z_i\} + B_{Y^*})$ is a covering of B_K since otherwise a point might be added to M. On the other hand, the sets $(\{z_i\}+\frac{1}{2}B_{Y^*})$ are disjoint for different i and contained in $B_K+\frac{1}{2}B_{Y^*}\subseteq\frac{3}{2}B_K$. Thus, comparing volumes,

$$L \operatorname{vol}_n(\tfrac{1}{2}B_{Y^*})\leqslant \operatorname{vol}_n(\tfrac{3}{2}B_K),$$
$$L^{1/n}\leqslant 3(\operatorname{vol}_n B_K/\operatorname{vol}_n B_{Y^*})^{1/n}\leqslant 3T_2(Y^*)$$

using Lemma 1.2. Hence $L\leqslant 2^{k-1}$ for $k=[bn]+2$. Thus $e_{[bn]+2}(j^{*-1})\leqslant 1$. Similarly, $e_{[an]+2}(i^{*-1})\leqslant 1$. This yields $e_{[cn]}(T)\leqslant e_n(T^*)$ by (1.3) with $c=\log_2(18\,T_2(X)\,T_2(Y^*))$, since $m\leqslant[cn]$ for $n\geqslant 2$.

(ii) We now drop the assumption that X and Y are n-dimensional and consider Banach spaces X and Y with X, Y^* of type 2 and $T\in L(X, Y)$ has $\operatorname{rank}(T)\leqslant n$. Consider the canonical factorization of T

$$T\colon X\xrightarrow{\pi} X/\operatorname{Ker} T\xrightarrow{\bar T}\operatorname{Im}(T)\xrightarrow{j} Y,$$

$\bar T$ induced by T, π projection, and j injection map. Since the entropy numbers are projective and injective up to a constant of 2, and $\bar T$ acts between $\leqslant n$-dimensional spaces, we find using (i)

$$\begin{aligned}e_{[cn]}(T)&=e_{[cn]}(j\bar T\pi)\leqslant 2e_{[cn]}(\bar T)\\&\leqslant 2e_n(\bar T^*)\leqslant 4e_n(\pi^*\bar T^* j^*)=4e_n(T^*),\end{aligned}$$

where $c=\operatorname{Log}_2(18\,T_2(X/\operatorname{Ker} T)\,T_2((\operatorname{Im} T)^*))$. Both spaces $X/\operatorname{Ker} T$ as well as $(\operatorname{Im} T)^*$ are quotient spaces of type 2 and thus

$$T_2(X/\operatorname{Ker} T)\leqslant T_2(X)\,\gamma(X),\qquad T_2((\operatorname{Im} T)^*)\leqslant T_2(Y^*)\,\gamma(Y^*).\quad\blacksquare$$

Remark. Without type 2 assumptions the following can be shown: There is some absolute constant a such that for all operators $T\in L(X, Y)$ of $\operatorname{rank}(T)\leqslant n$,

$$e_{an\ln(n+1)}(T)\leqslant 4e_n(T^*).$$

In the case of $\dim X=\dim Y=n$, one uses the Lewis maps $i\colon X\to l_2^n$, $j\colon l_2^n\to Y$ with $l(i^{-1})\,l^*(i)=n=l(j)\,l^*(j^{-1})$, cf. Lewis [9], to construct a Hilbert space factorization of T. The analog of Lemma 1.2 for i, j can be proved along the lines of Gordon and Reisner [6]. Estimates of $L^{1/n}$ similar to those in (i) now involve $\|T\|$, $\|T^{-1}\|$, which can be bounded by n, yielding the logarithmic factor.

LEMMA 1.4. *Let X and Y be Banach spaces such that X and Y^* are of type 2. Then for any $T\in L(X, Y)$ and $b:=T_2(X)\,T_2(Y^*)$*

$$c_n(T)\leqslant a_n(T)\leqslant bc_n(T),\qquad d_n(T)\leqslant a_n(T)\leqslant bd_n(T),\qquad n\in\mathbb{N}.\quad(1.4)$$

Proof. Let $\varepsilon > 0$ and choose a subspace $X_n \subset X$ of codimension $< n$ with

$$\|T|_{X_n}\| \leqslant (1+\varepsilon)\, c_n(T).$$

By Maurey's extension theorem [11] $T|_{X_n}$ has an extension to X, $S: X \to Y$ with $\|S\| \leqslant b\|T|_{X_n}\|$, $S|_{X_n} = T|_{X_n}$. Therefore, $T - S$ has rank $< n$ and

$$a_n(T) \leqslant \|T - (T-S)\| = \|S\| \leqslant (1+\varepsilon)\, bc_n(T).$$

The statement for the Kolmogorov numbers follows by duality. ▌

The following proposition is due to Carl [1].

PROPOSITION 1.5. *There is some absolute constant $a > 0$ such that for all Banach spaces X and Y with X, Y^* being of type 2, all $T \in L(X, Y)$ and all $n \in \mathbb{N}$,*

$$\left(\prod_{j=1}^{n} c_j(T)\right)^{1/n} \leqslant a T_2(X)\, T_2(Y^*)\, e_n(T).$$

COROLLARY 1.6. *Let X, Y be Banach spaces with X and Y^* of type 2. Then we have for all $T \in L(X, Y)$*

$$a_n(T) \leqslant d e_n(T),$$

where $d := a(T_2(X)\, T_2(Y^*))^2$, a from Proposition 1.5.

Proof. The result follows immediately from Lemma 1.4 and Proposition 1.5. ▌

Proof of Theorem 1.1. We reduce (1.1) to rank n operators by approximation. Let c be as in Lemma 1.3 and $T \in L(X, Y)$. For $\varepsilon > 0$, choose $T_n \in L(X, Y)$ of rank $(T_n) < n$ with $\|T - T_n\| \leqslant (1+\varepsilon)\, a_n(T)$. Then

$$e_{[cn]}(T) \leqslant \|T - T_n\| + e_{[cn]}(T_n) \leqslant (1+\varepsilon)\, a_n(T) + e_{[cn]}(T_n).$$

By Lemma 1.3, $e_{[cn]}(T_n) \leqslant 4e_n(T_n^*)$. The principle of local reflexivity yields $a_n(T) = a_n(T^*)$ for compact operators T (cf. Pietsch [13, 11.7.4]). For non-compact $T \in L(X, Y)$ at least $a_n(T) \leqslant 3a_n(T^*)$ can be shown. Hence,

$$\begin{aligned}
e_{[cn]}(T) &\leqslant 3(1+\varepsilon)\, a_n(T^*) + 4e_n(T_n^*) \\
&\leqslant 3(1+\varepsilon)\, a_n(T^*) + 4\|T^* - T_n^*\| + 4e_n(T^*) \\
&\leqslant 15(1+\varepsilon)\, a_n(T^*) + 4e_n(T^*) \\
&\leqslant (15(1+\varepsilon)\,(a(T_2(X)\, T_2(Y^*))^2 + 4)\, e_n(T^*)
\end{aligned}$$

where the last inequality follows from Corollary 1.6. The right side inequality in (1.1) follows by duality. ▌

Remark. By a modification of the proof of Theorem 1.1 and Lemma 1.3 one can show: For any $\varepsilon > 0$ there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geqslant n_0$

$$d^{-1} e_{[(1+\varepsilon)n]}(T) \leqslant e_n(T^*) \leqslant d e_{[(1-\varepsilon)n]}(T), \tag{1.5}$$

d being as in Theorem 1.1. Let $0 < \varepsilon < 1$. To prove the statement, one has to change m in (1.3) to $m = [an] + [bn] + [\alpha n] + 2$ for a suitable $\alpha > 3(a+b)/\varepsilon$, yielding

$$e_m(T) \leqslant 4 e_{[\alpha n]}(T^*)$$

for rank n operators T. Since $(a+b+\alpha)(n+1)/\alpha n$ is less than $(1+\varepsilon)$ for n larger than $3/\varepsilon$, standard arguments using the monotonicity of the entropy numbers will imply (1.5) for rank n operators. The approximation argument above remains unchanged using (1.5) instead of (1.2).

In the proof of Lemma 1.3 we used the entropy numbers of diagonal operators on Hilbert space. Pietsch [13] showed in 12.2.5. that an operator $D_\sigma: l_2 \to l_2$, $(x_n)_{n=1}^\infty \mapsto (\sigma_n x_n)_{n=1}^\infty$ has pth power summable entropy numbers iff $\sigma \in l_p$, without calculating $e_n(D_\sigma)$ explicitly. We now give an asymptotic formula for $e_n(D_\sigma)$ in sequence spaces.

PROPOSITION 1.7. *Let X be a real Banach space with a* 1*-unconditional basis $\{x_i\}_{i=1}^\infty$. Let $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant 0$ be decreasing and $D_\sigma: X \to X$ the diagonal operator induced by $x_i \mapsto \sigma_i x_i$. Then for all $k \in \mathbb{N}$*

$$\sup_{n \in \mathbb{N}} 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n} \leqslant e_{k+1}(D_\sigma) \leqslant 6 \sup_{n \in \mathbb{N}} 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n}.$$

Proof. For the left inequality, choose $n \in \mathbb{N}$. Let $X_n = \operatorname{span}\{x_i\}_{i=1}^n \subset X$ and $D_\sigma^n := D_{\sigma | X_n}: X_n \to X_n$. Then $e_{k+1}(D_\sigma) \geqslant e_{k+1}(D_\sigma^n)$ and with $B_n := B_{X_n}$ any covering $D_\sigma^n(B_n) \subseteq \bigcup_{j=1}^{2^k} (\{\tilde{x}_j\} + \delta B_n)$ with $\delta > 0$ yields the volume estimate

$$(\sigma_1 \cdots \sigma_n) \operatorname{vol}_n(B_n) = \operatorname{vol}_n(D_\sigma^n(B_n))$$
$$\leqslant 2^k \delta^n \operatorname{vol}_n(B_n),$$
$$2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n} \leqslant \delta.$$

Thus,

$$e_{k+1}(D_\sigma) \geqslant e_{k+1}(D_\sigma^n) \geqslant 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n}.$$

To prove the right-hand side inequality put $\delta := 8 \sup_{l \in \mathbb{N}} 2^{-k/l}(\sigma_1 \cdots \sigma_l)^{1/l}$ and, for a given $\varepsilon > 0$, choose $n \in \mathbb{N}$ with

$$\delta \leqslant (8 + \varepsilon)\, 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n}.$$

If all $\sigma_1,\dots,\sigma_n$ are $\geqslant \delta/4$, let $r:=n$. Otherwise, determine $r \in \mathbb{N}$ by $\sigma_{r+1} < \delta/4 \leqslant \sigma_r$. Using similar notation as above, choose a maximal system of points $\{\tilde{x}_j\}_{j=1}^{L}$ in $D_\sigma^r(B_r)$ with

$$\|\tilde{x}_i - \tilde{x}_j\|_{X_r} > \delta/2, \qquad 1 \leqslant i \neq j \leqslant L.$$

Then $D_\sigma^r(B_r) \subseteq \bigcup_{j=1}^{L} (\{\tilde{x}_j\} + \delta/2 B_r)$ in view of the maximality of the system $\{\tilde{x}_j\}_{j=1}^{L}$. Letting $B = B_X$, any element in $D_\sigma(B)$ can be written as $D_\sigma x = \sum_{i=1}^{\infty} \sigma_i \alpha_i x_i$ for $x = \sum_{i=1}^{\infty} \alpha_i x_i \in B$. Since $\{x_i\}_{i=1}^{\infty}$ is 1-unconditional, we find

$$D_\sigma x = \sum_{i=1}^{r} \sigma_i \alpha_i x_i + \sum_{i=r+1}^{\infty} \sigma_i \alpha_i x_i \in D_\sigma^r(B_r) + \delta/4\, B.$$

Thus, $D_\sigma(B) \subseteq \bigcup_{j=1}^{L} (\{\tilde{x}_j\} + \frac{3}{4}\delta B)$. Hence, $e_{[\log_2 L]+2}(D_\sigma) \leqslant \frac{3}{4}\delta$.

To estimate L, note that

$$\bigcup_{j=1}^{L} (\{\tilde{x}_j\} + \delta/4\, B_r) \subseteq D_\sigma^r(B_r) + \delta/4\, B_r \subseteq 2D_\sigma^r(B_r)$$

is a disjoint union sitting in $2D_\sigma^r(B_r)$. Hence,

$$L(\delta/4)^r \operatorname{vol}_r(B_r) \leqslant 2^r(\sigma_1 \cdots \sigma_r) \operatorname{vol}_r(B_r).$$

By choice of $n \in \mathbb{N}$, $(\sigma_1 \cdots \sigma_r) \leqslant 2^k (\delta/8)^r$. Therefore,

$$L \leqslant (8/\delta)^r (\sigma_1 \cdots \sigma_r) \leqslant 2^k,$$

implying $e_{k+1}(D_\sigma) \leqslant \frac{3}{4}\delta$. Letting $\varepsilon \to 0$ ends the proof. ∎

Corollary 1.8. *Let X be a real Banach space with* 1*-unconditional basis* $\{x_i\}_{i=1}^{\infty}$. *Let* $\sigma_1 \geqslant \sigma_2 \geqslant \cdots > 0$ *and* $D_\sigma : X \to X$ *the operator given by* $x_i \mapsto \sigma_i x_i$. *Then*

$$(e_k(D_\sigma))_{k=1}^{\infty} \in l_p \Leftrightarrow \sigma \in l_p.$$

Proof. Obviously, $e_{k+1}(D_\sigma) \geqslant \sigma_k/2$ yields the implication "$\Rightarrow$." Assume now $\sigma \in l_p$. Let $0 < q < p$ and $\mu \in l_q$ be positive decreasing. Then $\mu_n \leqslant \|\mu\|_q n^{-1/q}$ for all $n \in \mathbb{N}$. Proposition 1.7 gives

$$e_k(D_\mu) \leqslant c_q \|\mu\|_q\, k^{-1/q}, \qquad k \in \mathbb{N},$$

with c_q depending only on q. Apply this to the sequence $\mu = (\sigma_1 \cdots \sigma_k, 0, \dots)$ to get $e_k(D_\sigma^k) \leqslant c_q \left(\sum_{j=1}^{k} \sigma_j^q / k\right)^{1/q}$. Hence,

$$e_k(D_\sigma) \leqslant \|D_\sigma - D_\sigma^k\| + e_k(D_\sigma^k) \leqslant \sigma_{k+1} + e_k(D_\sigma^k)$$
$$\leqslant (2+c_q)\left(\sum_{j=1}^{k} \sigma_j^q/k\right)^{1/q},$$
$$\left(\sum_{k=1}^{\infty} e_k(D_\sigma)^p\right)^{1/p} \leqslant (2+c_q)\left(\sum_{k=1}^{\infty}\left(\frac{\sum_{j=1}^{k}\sigma_j^q}{k}\right)^{p/q}\right)^{1/p} \leqslant d_{p,q}\left(\sum_{k=1}^{\infty}\sigma_k^p\right)^{1/p} < \infty,$$

where the last estimate is Hardy's inequality, cf. [7]. ∎

Remark. Similar statements hold in the case of complex Banach spaces if the exponent k/n in $2^{-k/n}$ is changed to $k/(2n)$; one has to use the $(2n)$-dimensional volume in $\mathbb{C}^n = \mathbb{R}^{2n}$.

2. Approximation, Gelfand and Kolmogorov Numbers

In this section we derive some general estimates for s-numbers using probabilistic methods. $\{g_{i,j}\}_{i,j=1}^n$, $\{h_{i,j}\}_{i,j=1}^n$, and $\{g_i\}_{i=1}^n$ will denote standard independent Gaussian variables. For $\{x_i\}_{i=1}^n \subset X$ we let

$$\varepsilon_2(\{x_i\}_{i=1}^n) := \max_{\|t\|_2=1}\left\|\sum_{i=1}^n t_i x_i\right\|.$$

THEOREM 2.1. *Let* $T \in L(X, Y)$ *with* $T = \sum_{i=1}^n x_i^* \otimes y_i$, $\{x_i^*\} \subset X^*$, $\{y_i\} \subset Y$. *Then for* $k < n$,

$$a_{k+1}(T) \leqslant \frac{4}{\sqrt{k}}\left\{\varepsilon_2(\{x_i^*\}_{i=1}^n)\,\mathbb{E}\left\|\sum_{i=1}^n g_i y_i\right\| + \varepsilon_2(\{y_i\}_{i=1}^n)\,\mathbb{E}\left\|\sum_{i=1}^n g_i x_i^*\right\| + k^{-1/2}\,\mathbb{E}\left\|\sum_{i=1}^n g_i x_i^*\right\|\,\mathbb{E}\left\|\sum_{i=1}^n g_i y_i\right\|\right\}.$$

For the proof we need the following inequality of Chevet [2]:

$$\mathbb{E}\left\|\sum_{i,j=1}^n g_{i,j} x_i^* \otimes y_j\right\| \leqslant \varepsilon_2(\{x_i^*\}_{i=1}^n)\,\mathbb{E}\left\|\sum_{i=1}^n g_i y_i\right\| + \varepsilon_2(\{y_i\}_{i=1}^n)\,E\left\|\sum_{i=1}^n g_i x_i^*\right\| \tag{2.1}$$

(Chevet's proof included a multiplicative constant $\sqrt{2}$ on the right-hand side which was improved to 1 in [5]).

Proof. Without loss of generality we may assume that $\{y_i\}_{i=1}^n$ are independent. Therefore, there is a biorthogonal sequence $\{y_i^*\}_{i=1}^n \subset Y^*$. Let $G(\omega)\colon X \to Y$, $P\colon Y \to Y$, and $G^*(\omega)\colon Y \to Y$ be defined by

$$G(\omega) = k^{-1/2} \sum_{i,j=1}^{n} g_{i,j}(\omega)\, x_i^* \otimes y_j$$

$$P = \sum_{j=1}^{k} y_j^* \otimes y_j$$

$$G^*(\omega) = k^{-1/2} \sum_{l,j=1}^{n} g_{l,j}(\omega)\, y_j^* \otimes y_l$$

and let H and H^* be defined as G and G^*, respectively, with variables $h_{i,j}$ on a different probability space. Obviously $\operatorname{rank}(G^*PG) \leqslant k$, and easy calculation shows that $T = \mathbb{E}(H^*PH)$. Therefore,

$$\begin{aligned} a_{k+1}(T) &\leqslant \mathbb{E}_\omega \|T - G^*(\omega)\, PG(\omega)\| \\ &= \mathbb{E}_\omega \|\mathbb{E}_{\omega'}(H^*(\omega')\, PH(\omega')) - G^*(\omega)\, PG(\omega)\| \\ &\leqslant \mathbb{E}_\omega \mathbb{E}_{\omega'} \|H^*(\omega')\, PH(\omega') - G^*(\omega)\, PG(\omega)\| \\ &= \mathbb{E}_\omega \mathbb{E}_{\omega'} \left\| 2\left(\frac{H^*(\omega') + G^*(\omega)}{\sqrt{2}}\right) P \left(\frac{H(\omega') - G(\omega)}{\sqrt{2}}\right) \right. \\ &\qquad \left. - G^*(\omega)\, PH(\omega') + H^*(\omega')\, PG(\omega) \right\|. \end{aligned}$$

$(H^*(\omega') + G^*(\omega))/\sqrt{2}$ and $(H(\omega') - G(\omega))/\sqrt{2}$ have the same distribution as $H^*(\omega')$ and $G(\omega)$, respectively. Moreover, they are independent, hence we obtain

$$a_{k+1}(T) \leqslant 4\mathbb{E}_{\omega'} \mathbb{E}_\omega \|H^*(\omega')\, PG(\omega)\|.$$

Fix ω' and let $H^* = H^*(\omega')$. Then by (2.1) we get

$$\begin{aligned} \mathbb{E}_\omega \|H^* PG(\omega)\| &= \mathbb{E}_\omega \left\| k^{-1/2} \sum_{i=1}^{n} \sum_{j=1}^{k} g_{i,j}(\omega)\, x_i^* \otimes H^*(y_j) \right\| \\ &\leqslant k^{-1/2} \left\{ \varepsilon_2(\{x_i^*\}_{i=1}^n)\, \mathbb{E}_\omega \left\| \sum_{j=1}^{k} g_j(\omega)\, H^*(y_j) \right\| \right. \\ &\qquad \left. + \varepsilon_2(\{H^*(y_j)\}_{j=1}^k)\, \mathbb{E}_\omega \left\| \sum_{i=1}^{n} g_i(\omega)\, x_i^* \right\| \right\}. \qquad (2.2) \end{aligned}$$

Now we integrate over ω'. $\varepsilon_2(\{H^*(\omega')\, y_j\}_{j=1}^k)$ is the norm of the operator

$$k^{-1/2} \sum_{i=1}^{n} \sum_{j=1}^{k} h_{i,j}(\omega')\, e_j \otimes y_i$$

mapping l_2^k to Y, hence by inequality (2.1) it follows that

$$\begin{aligned}
&\mathbb{E}_{\omega'}\varepsilon_2(\{H^*(\omega')y_j\}_{j=1}^k)\\
&\leqslant k^{-1/2}\left\{\varepsilon_2(\{e_j\}_{j=1}^k)\,\mathbb{E}_{\omega'}\left\|\sum_{i=1}^n g_i(\omega')y_i\right\|\right.\\
&\qquad\left.+\varepsilon_2(\{y_i\}_{i=1}^n)\,\mathbb{E}_{\omega'}\left\|\sum_{j=1}^k g_j(\omega')e_j\right\|\right\}\\
&\leqslant k^{-1/2}\left\{E_{\omega'}\left\|\sum_{i=1}^n g_i(\omega')y_i\right\|+\sqrt{k}\,\varepsilon_2(\{y_i\}_{i=1}^n)\right\}. \qquad (2.3)
\end{aligned}$$

Since $\{\sum_{j=1}^k t_j g_{i,j}\}_{i=1}^n$ is equivalent to $\{(\sum_{j=1}^k |t_j|^2)^{1/2} g_i\}_{i=1}^n$, we have

$$\begin{aligned}
\mathbb{E}_\omega\mathbb{E}_{\omega'}\left\|\sum_{j=1}^k g_j(\omega)\,H^*(\omega')(y_j)\right\| &= k^{-1/2}\mathbb{E}_\omega\mathbb{E}_{\omega'}\left\|\sum_{i=1}^n\sum_{j=1}^k g_j(\omega)\,g_{i,j}(\omega')\,y_i\right\|\\
&= k^{-1/2}\mathbb{E}_\omega\left(\sum_{j=1}^k |g_j(\omega)|^2\right)^{1/2} E_{\omega'}\left\|\sum_{i=1}^n g_i(\omega')y_i\right\|\\
&\leqslant E_{\omega'}\left\|\sum_{i=1}^n g_i(\omega')y_i\right\|.
\end{aligned}$$

Theorem 2.1 now follows from this, (2.2), and (2.3). ∎

PROPOSITION 2.2. *Let* $T\in L(X,Y)$, $\{x_i\}_{i=1}^n$ *be a basis of* X *and* $y_i=T(x_i)$, $i=1,2,\dots,n$. *Then we have for* $k<n$

$$c_{k+1}(T)\leqslant\frac{\mathbb{E}\|\sum_{i=1}^n g_i y_i\|+\sqrt{n-k}\,\varepsilon_2(\{y_i\}_{i=1}^n)}{\mathbb{E}\|\sum_{i=1}^n g_i x_i\|-\sqrt{n-k}\,\varepsilon_2(\{x_i\}_{i=1}^n)}.$$

Proof. We shall need the following inequality due to Gordon [5]

$$\mathbb{E}\left\|\sum_{j=1}^n g_j x_j\right\|-\sqrt{k}\,\varepsilon_2(\{x_j\}_{j=1}^n)\leqslant\mathbb{E}\left(\min_{\|t\|_2=1}\left\|\sum_{i=1}^k\sum_{j=1}^n g_{i,j}(\omega)\,t_i y_j\right\|\right). \qquad (2.4)$$

Let

$$L_\omega=\operatorname{span}\left\{\sum_{j=1}^n g_{i,j}(\omega)\,x_j\right\}_{i=1}^{n-k},$$

then $\dim(L_\omega)=n-k$ except on a set of measure 0. Therefore we have a.e.

$$\begin{aligned}
c_{k+1}(T)\leqslant\|T|_{L_\omega}\| &= \sup_{\|t\|_2=1}\frac{\|\sum_{i=1}^{n-k}t_i\sum_{j=1}^n g_{i,j}(\omega)\,y_j\|}{\|\sum_{i=1}^{n-k}t_i\sum_{j=1}^n g_{i,j}(\omega)\,x_j\|}\\
&\leqslant\frac{\sup_{\|t\|_2=1}\|\sum_{i=1}^{n-k}t_i\sum_{j=1}^n g_{i,j}(\omega)\,y_j\|}{\inf_{\|t\|_2=1}\|\sum_{i=1}^{n-k}t_i\sum_{j=1}^n g_{i,j}(\omega)\,x_j\|}.
\end{aligned}$$

We multiply by the denominator and integrate with respect to ω. The left-hand side can then be estimated from below by (2.4) and the right-hand side

$$\mathbb{E}\left\|\sum_{i=1}^{n-k}\sum_{j=1}^{n} g_{i,j}(\omega)\, e_i \otimes y_j\right\|_{L(l_2^{n-k},\, Y)}$$

from above by (2.1). This yields Proposition 2.2. ∎

PROPOSITION 2.3. *Let $S \in L(X, Y)$ and $y_i \in S(B_X)$, $i = 1,\dots, n$, be linearly independent. Moreover, consider $I \in L(Y, l_2^n)$ with $I(y_i) = e_i$, $i = 1,\dots, n$. Then one has for $k < n$*

$$d_{k+1}(S) \geqslant \frac{n-\sqrt{kn}}{\sqrt{kn}\,\|I\| + \sum_{i=1}^{n} \|I^*(e_i^*)\|}.$$

Proof. Given $\varepsilon > 0$, choose a subspace L of Y of dimension $\leqslant k$ and vectors $w_i \in L$, $i = 1,\dots, n$, in $S(B_X)$ such that

$$(1+\varepsilon)\, d_{k+1}(S) \geqslant \|y_i - w_i\|$$

for all $i = 1,\dots, n$. We have

$$\begin{aligned}(1+\varepsilon)\, d_{k+1}(S) \sum_{i=1}^{n} \|I^*(e_i^*)\| &\geqslant \sum_{i=1}^{n} \|y_i - w_i\|\, \|I^*(e_i^*)\| \\ &\geqslant n - \sum_{i=1}^{n} \langle w_i, I^*(e_i^*)\rangle \\ &= n - \operatorname{tr}\left(\sum_{i=1}^{n} I(w_i) \otimes e_i^*\right).\end{aligned}$$

Since $\dim L \leqslant k$, we obtain

$$\begin{aligned}\operatorname{tr}\left(\sum_{i=1}^{n} I(w_i)\otimes e_i^*\right) &\leqslant \sum_{l=1}^{k} s_l\left(\sum_{i=1}^{n} I(w_i)\otimes e_i^*\right) \\ &\leqslant \sqrt{k}\,\pi_2\left(\sum_{i=1}^{n} I(w_i)\otimes e_i^*\right) \\ &= \sqrt{k}\left(\sum_{i=1}^{n} \|I(w_i)\|_2^2\right)^{1/2} \\ &\leqslant \sqrt{k}\left\{\left(\sum_{i=1}^{n} \|I(w_i) - I(y_i)\|_2^2\right)^{1/2} + \left(\sum_{i=1}^{n} \|e_i\|_2^2\right)^{1/2}\right\} \\ &\leqslant \sqrt{k}\,\{\|I\|\,\sqrt{n}\,(1+\varepsilon)\, d_{k+1}(S) + \sqrt{n}\}.\end{aligned}$$

Therefore, we get

$$(1+\varepsilon)\, d_{k+1}(S) \left\{ \sqrt{kn}\, \|I\| + \sum_{i=1}^{n} \|I^*(e_i^*)\| \right\} \geqslant n - \sqrt{kn}. \quad ▮$$

PROPOSITION 2.4. *Assume that* $\dim(X) = \dim(Y) = n$ *and that* $S \in L(X, Y)$ *is an invertible operator. Then we have for all* $k < n$

$$c_{k+1}(S) \geqslant \gamma_2(S^{-1})^{-1} \inf\{d(E, l_2^{n-k}) \mid E \subseteq Y, \quad \dim(E) = n-k\}.$$

Proof. There exist $A \in L(Y, l_2^n)$, $B \in L(l_2^n, X)$ such that $\gamma_2(S^{-1}) = \|A\|\,\|B\|$ and $S^{-1} = BA$. Let

$$\alpha = \|A\|\, \{\inf\{d(E, l_2^{n-k}) \mid E \subseteq Y, \quad \dim(E) = n-k\}\}^{-1}.$$

There is a subspace L of X such that $c_{k+1}(S) = \|S|_L\|$, where $\dim(L) \geqslant n-k$. We claim that there exists a vector $x_0 \in L$ such that

$$\|Sx_0\| = 1 \qquad \text{and} \qquad \|B^{-1}x_0\| \leqslant \alpha \tag{2.5}$$

since, otherwise, we have for all $x \neq 0$, $x \in L$, $\|B^{-1}x\| > \alpha \|Sx\|$, therefore

$$\|B^{-1}x\| > \alpha\|Sx\| = \alpha\|A^{-1}B^{-1}x\| \geqslant \alpha\|A\|^{-1}\|B^{-1}x\|,$$

which implies that $d(S(L), l_2^{n-k}) < \|A\|\,\alpha^{-1}$; but this contradicts the definition of α. Using (2.5) we obtain

$$c_{k+1}(S) \geqslant \|S|_L\| \geqslant \|Sx_0\|/\|x_0\| \geqslant (\alpha\|B\|)^{-1},$$

which ends the proof. ▮

Remark. Let s be the largest dimension of a 2-isomorph of l_2^s in the space Y. Then, by using Theorem 5.2 in [3] we get

$$\sqrt{s} \geqslant c(n-k)^{1/2}/\inf\{d(E, l_2^{n-k}) \mid E \subset Y\}$$

for some absolute constant $c > 0$. Hence we obtain from Proposition 2.4

$$c_{k+1}(S) \geqslant c(n-k)^{1/2}\, s^{-1/2} \gamma_2(S^{-1})^{-1}. \tag{2.6}$$

3. APPLICATIONS

In this section we apply the previous results to derive asymptotic estimates for the approximation numbers of the identity maps between general symmetric spaces as well as the spaces of operators c_p^n and special tensor product spaces. The first results are direct corollaries of Section 2.

COROLLARY 3.1. *Let* $\{x_i\}_{i=1}^n \subset Y$ *and* $\{y_i\}_{i=1}^n \subset Y$ *be normalized* 1-*unconditional bases of* X *and* Y, *respectively. Define* $id_{X,Y} \in L(X, Y)$ *by* $id_{X,Y}(x_i) = y_i$, $i = 1,\ldots, n$. *Then we have for* $k < n$,

$$a_{k+1}(id_{X,Y}) \leqslant \begin{cases} c; & \text{if } 1 \leqslant k \leqslant \max\{\lambda_{X^*}^2(n), \lambda_Y^2(n)\} \\ ck^{-1/2}\max\{\lambda_{X^*}(n), \lambda_Y(n)\}; & \text{if } \max\{\lambda_{X^*}^2(n), \lambda_Y^2(n)\} \leqslant k, \end{cases}$$

where $c = c(T_2(X^*), T_2(Y))$ *is a constant that depends only on the type* 2-*constants of* X^* *and* Y.

Proof. This follows from Theorem 2.1 since

$$\varepsilon_2(\{x_i^*\}_{i=1}^n) = \max_{\|t\|_2=1} \left\| \sum_{i=1}^n t_i x_i^* \right\| = \max_{\|t\|_2=1} \operatorname*{Ave}_{\pm} \left\| \sum_{i=1}^n \pm t_i x_i^* \right\| \leqslant \sqrt{\pi/2}\, T_2(X^*)$$

and

$$\mathbb{E} \left\| \sum_{i=1}^n g_i y_i \right\| \leqslant c(T_2(Y))\, \mathbb{E} \left\| \sum_{i=1}^n r_i y_i \right\| = c(T_2(Y)) \left\| \sum_{i=1}^n y_i \right\|,$$

and similar inequalities hold if we interchange $\{x_i^*\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$. Also we have to observe that

$$\|id_{X,Y}\| \leqslant \max_{\|t\|_2=1} \left\| \sum_{i=1}^n t_i y_i \right\| \Big/ \min_{\|t\|_2=1} \left\| \sum_{i=1}^n t_i x_i \right\|$$
$$= \varepsilon_2(\{y_i\}_{i=1}^n)\, \varepsilon_2(\{x_i^*\}_{i=1}^n) \leqslant \frac{\pi}{2} T_2(X^*)\, T_2(Y). \quad \blacksquare$$

COROLLARY 3.2. *Let* $\{x_i\}_{i=1}^n$ *and* $\{y_i\}_{i=1}^n$ *be normalized* 1-*symmetric bases of* X *and* Y. *Let* $id_{X,Y} \in L(X, Y)$ *be defined as above. Then we have*

$$d_{k+1}(id_{X,Y}) \geqslant \begin{cases} c; & \text{if } 1 \leqslant k \leqslant \lambda_Y^2(n) \\ c\lambda_Y(n)/\sqrt{k}; & \text{if } \lambda_Y^2(n) \leqslant k \leqslant n/2, \end{cases}$$

where $c = c(T_2(Y))$ *is a constant that depends only on the type* 2-*constants of* Y.

Proof. We apply Proposition 2.3. Since $\{y_i\}_{i=1}^n$ is a 1-symmetric basis,

$$\sum_{i=1}^n \|I^*(e_i^*)\| = \sum_{i=1}^n \|y_i^*\| = n.$$

Using the fact that a Banach lattice of type 2 is 2-convex and applying the argument in the proof of Proposition 3.7 [8] we find

$$\|I\| \leqslant C(T_2(Y))\sqrt{n}\Big/\Big\|\sum_{i=1}^{n} y_i\Big\|.$$

Thus we obtain

$$d_{k+1}(id_{X,Y}) \geqslant (n-\sqrt{kn})\Big/\left(C(T_2(Y))\, n\sqrt{k}\,\Big\|\sum_{i=1}^{n} y_i\Big\|^{-1} + n\right). \quad \blacksquare$$

The previous two corollaries together yield

COROLLARY 3.3. *Let* $\{x_i\}_{i=1}^n$ *and* $\{y_i\}_{i=1}^n$ *be normalized* 1*-symmetric bases of* X *and* Y. *Let* $id_{X,Y} \in L(X, Y)$ *map* x_i *to* y_i, $i = 1,..., n$. *Then we have for all* $1 \leqslant k \leqslant n/2$

$$c^{-1} a_k(id_{X,Y}) \leqslant \max\{d_k(id_{X,Y}), c_k(id_{X,Y})\} \leqslant a_k(id_{X,Y}),$$

where $c = c(T_2(X^*), T_2(Y))$ *depends only on the type* 2*-constants of* X^* *and* Y.

Remark. For $X = l_p^n$, $Y = l_q^n$, $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ and $k \leqslant n/2$ we obtain the results of Gluskin [4] again. For some further applications in the case $k \geqslant n/2$ we need Gluskin's method.

DEFINITION. Let $X = (\mathbb{R}^n, \|\cdot\|)$ be n-dimensional and $N \in \mathbb{N}$, $a > 0$. We say that X has N *a-essential extreme points* iff there are elements $\lambda_1,..., \lambda_N \in B_{l_2^n}$ with

$$aB_X \subseteq \Gamma(\lambda_1,..., \lambda_N) \subseteq B_{l_2^n},$$

where $\Gamma(A)$ denotes the convex hull of a given set A. Gluskin [4] used this concept to prove

THEOREM 3.4. *Let* $\theta > 0$, $a > 0$, $n, N \in \mathbb{N}$. *Assume that* $X = (\mathbb{R}^n, \|\cdot\|_X)$ *and* $Y = (\mathbb{R}^n, \|\cdot\|_Y)$ *have* N *a-essential extreme points with* $N < (\frac{1}{16}) \exp(\theta^2 n/4)$. *Then there is* $c = c(a)$ *depending only on a such that*

$$a_k(Id\colon X \to Y^*) \leqslant c(\theta\sqrt{(n\text{-}k)/k} + \theta^2(n/k)).$$

Gluskin applied this to $X = Y = l_p^n$, $1 \leqslant p \leqslant 2$. We consider $X = Y = l_p^n \otimes_\pi l_p^n$, which is the dual of $X^* = Y^* = l_{p'}^n \otimes_\varepsilon l_{p'}^n = L(l_p^n, l_{p'}^n)$, where $1/p + 1/p' = 1$.

LEMMA 3.5. *Let* $1 < p \leqslant 2$. *Then there are* $a_p, b_p > 0$ *such that* $l_p^n \otimes_\pi l_p^n$ *has* N a_p*-essential extreme points for some* $N \in \mathbb{N}$ *with*

$$N \leqslant \exp(b_p n^{2/p'}).$$

Proof. By [4], l_p^n has $N \leqslant \exp(\beta_p n^{2/p'})$ α_p-essential extreme points for some $\alpha_p, \beta_p > 0$ depending only on p. Denote them by $\lambda_1, ..., \lambda_N$. Then $\|\lambda_i \otimes \lambda_j\|_{l_2^{n^2}} \leqslant 1$ and

$$\alpha_p^2 B_{l_p^n \otimes_\pi l_p^n} \subseteq \alpha_p^2 \Gamma(x \otimes y \mid \|x\|_{l_p^n} = \|y\|_{l_p^n} = 1)$$
$$\subseteq \Gamma(\lambda_i \otimes \lambda_j \mid i, j = 1, ..., N) \subseteq B_{l_2^{n^2}}.$$

Since $N^2 \leqslant \exp(2\beta_p n^{2/p'})$, this proves the lemma with $a_p = \alpha_p^2$, $b_p = 2\beta_p$. ▮

PROPOSITION 3.6. *Let $1 < p \leqslant 2$. Then there are constants $a_p, b_p > 0$ such that c_p^n has N a_p-essential extreme points for some $N \in \mathbb{N}$ with*

$$N \leqslant \exp(b_p n^{2/p'+1} (\ln n)).$$

Proof. By Gluskin [4] there are $\alpha_p, \beta_p > 0$ such that l_p^n has M α_p-essential extreme points for $M \leqslant \exp(\beta_p n^{2/p'})$. Thus there is $\Lambda := \{\lambda^1, ..., \lambda^M\} \subseteq B_{l_2^n}$ such that

$$\alpha_p B_{l_p^n} \subseteq \Gamma(\lambda^1, ..., \lambda^M) \subseteq B_{l_2^n}. \tag{3.1}$$

Moreover, the proof of Lemma 1 of [4] and the remarks before it show that the λ^i can be chosen in such a way as to have support $S_i := \{j \in \{1, ..., n\} \mid \lambda_j^i \neq 0\}$ of cardinality

$$|S_i| \leqslant [n^{2/p'}] + 1.$$

Let $\delta = n^{-2}$ and choose a δ-net Δ in $B_{l_2^n}$ of cardinality

$$L := |\Delta| \leqslant (1 + 2/\delta)^n \leqslant \exp(2n \ln 2n).$$

Let

$$\Sigma := \left\{ T \in L(l_2^n, l_2^n) \mid T = \sum_{k=1}^n \lambda_k x_k \otimes y_k \quad \text{with} \quad \lambda = (\lambda_k)_{k=1}^n \in \Lambda, \right.$$
$$\left. x_k, y_k \in \Delta \quad \text{and} \quad c_2(T) \leqslant 2 \right\}.$$

Since for each i, $|S_i| \leqslant [n^{2/p'}] + 1$, the number of different elements in Σ can be estimated as

$$|\Sigma| \leqslant |\Delta|^{2([n^{2/p'}]+1)} |\Lambda|$$
$$\leqslant \exp(4n(\ln 2n)(n^{2/p'} + 1) + \beta_p n^{2/p'}) \leqslant \exp(b_p n^{2/p'+1} \ln n)$$

for some $b_p > 0$. We claim that for $n \geqslant 4\alpha_p^{-1}$

$$(2\alpha_p)^{-1} B_{c_p^n} \subseteq \Gamma(T \mid T \in \Sigma) \subseteq 2B_{c_2^n} = 2B_{l_2^{n^2}},$$

which would prove the proposition. The right inclusion follows from the definition of Σ. To prove the left inclusion, let $T \in c_p^n$ with $c_p(T) = 1$. Then there are $\lambda \in l_p^n$, $\|\lambda\|_p = 1$, and orthonormal systems $(u_k)_{k=1}^n$, $(v_k)_{k=1}^n \subset l_2^n$ with

$$T = \sum_{k=1}^{n} \lambda_k u_k \otimes v_k.$$

Choose $x_k, y_k \in \Delta$ with $\|u_k - x_k\|_{l_2^n} \leqslant n^{-2}$, $\|v_k - y_k\|_{l_2^n} \leqslant n^{-2}$. Further, by (3.1), there are $(a_i)_{i=1}^M \subseteq R^+$ with $\sum_{i=1}^M a_i \leqslant \alpha_p^{-1}$ such that $\lambda = \sum_{i=1}^M a_i \lambda^i$. Let

$$T_i := \sum_{k=1}^{n} \lambda_k^i u_k \otimes v_k, \qquad S_i := \sum_{k=1}^{n} \lambda_k^i x_k \otimes y_k.$$

Then $T = \sum_{k=1}^n a_i T_i$ and elementary estimates show

$$c_2(T_i - S_i) \leqslant c_p(T_i - S_i) \leqslant 2n^{-1},$$
$$c_2(S_i) \leqslant c_2(T_i) + c_2(T_i - S_i) \leqslant \|\lambda^i\|_2 + 2/n \leqslant 2 \qquad (n \geqslant 2).$$

Hence, $S_i \in \Sigma$ and $S := \sum_{k=1}^n a_i S_i \in \alpha_p^{-1} \Gamma(\Sigma)$ provides an approximation of T with

$$c_p(T - S) \leqslant \sum_{k=1}^{n} a_i c_p(T_i - S_i) \leqslant 2\alpha_p^{-1}/n \leqslant \tfrac{1}{2}$$

for $n \geqslant 4\alpha_p^{-1}$. Now let $T^1 := T - S$ and repeat the argument with T^1 instead of T. This yields $S^1 \in \frac{1}{2}\alpha_p^{-1}\Gamma(\Sigma)$ with

$$c_p(T^1 - S^1) \leqslant \tfrac{1}{4}.$$

Continuing this with $T^2 := T^1 - S^1$, we find $S^l \in (1/2^l)\, \alpha_p^{-1} \Gamma(\Sigma)$ with

$$c_p(T^l - S^l) \leqslant 1/2^{l+1}, \qquad T^{l+1} := T^l - S^l.$$

This gives with $S^0 := S$

$$T = S^0 + T^1 = S^0 + S^1 + T_2 = \cdots = \sum_{l=0}^{\infty} S^l \in 2\alpha_p^{-1}\Gamma(\Sigma),$$

which is what we claimed. ∎

Proposition 3.7. *Let $1 < p < 2$. Then with constants independent of $1 \leqslant k < n^2$, but depending on p,*

$$a_k(Id\colon c_p^n \to c_{p'}^n) \sim \begin{cases} 1; & \text{if } \ 1 \leqslant k \leqslant [n^{3-2/p}] \\ n^{1/p' + 1/2}/\sqrt{k}; & \text{if } \ [n^{3-2/p}] < k \leqslant n^2/2. \end{cases}$$

For $k > n^2/2$ *one has at least*: *If* $k \leqslant n^2 - b_p n^{3-2/p}(\ln n)$ *then* $d_p^{-1} \sqrt{n^2-k}\, n^{-(1/2)-1/p} \leqslant a_k(Id\colon\ c_p^n \to c_{p'}^n) \leqslant d_p \sqrt{n^2-k}\, n^{-(1/2)-1/p} \sqrt{\ln n}$, *and if* $n^2 - b_p n^{3-2/p}(\ln n) \leqslant k < n^2$ *then* $\max\{n^{1-2/p}, d_p^{-1}\sqrt{n^2-k}\, n^{-(1/2)-1/p}\} \leqslant a_k(Id\colon c_p^n \to c_{p'}^n) \leqslant d_p n^{1-2/p}(\ln n)$.

Proof. If $1 \leqslant k \leqslant n^2/2$, the upper estimate follows from the obvious fact $a_k \leqslant 1$ in the case $k \leqslant [n^{3-2/p}]$, and when $[n^{3-2/p}] < k < n^2/2$ from Theorem 2.1, where we use the estimate $d(c_{p'}^n, c_\infty^n) \leqslant n^{1/p'}$ and inequality (2.1) to obtain

$$\mathbb{E}\left\| \sum_{i,j=1}^{n} g_{i,j} e_i \otimes e_j \right\|_{c_{p'}^n} \leqslant n^{1/p'} \mathbb{E}\left\| \sum_{i,j=1}^{n} g_{i,j} e_i \otimes e_j \right\|_{c_\infty^n} \leqslant d_p n^{1/p'+1/2}.$$

If $n^2/2 < k < n$, the upper estimates are derived from Gluskin's Theorem 3.4 and Proposition 3.6. The lower estimates are a consequence of Proposition 2.3 in the case $k \leqslant n^2/2$, and of Proposition 2.4 for $k > n^2/2$. To apply Proposition 2.4 one has to know that largest 2-Hilbertian subspace in $c_{p'}^n$ has dimension of the order $n^{3-2/p}$ (see Example 3.3 [3]). For $k > n^2/2$ we also use the estimate $a_k(Id) \geqslant \|Id^{-1}\|^{-1} = n^{1-2/p}$. ∎

Remark. Probably the logarithmic terms for $k > n^2/2$ are not necessary. The approximation numbers $a_k(Id\colon c_p^n \to c_q^n)$ for $1 < p \leqslant 2 \leqslant q < \infty$ can be derived in a similar way, as in the case of l_p^n the case $p' = q$ is the essential one.

When $p = 1$ we have

PROPOSITION 3.8. *For* $1 \leqslant k < n^2$,

$$a_k(Id\colon c_1^n \to c_\infty^n) \sim \begin{cases} 1; & \text{if } 1 \leqslant k \leqslant n \\ \sqrt{n/k}; & \text{if } n \leqslant k < n^2/2 \\ \sqrt{n^2-k}\, n^{-3/2}; & \text{if } n^2/2 \leqslant k < n^2 - n \\ n^{-1}; & \text{if } n^2 - n < k < n^2. \end{cases}$$

Proof. We apply the same proof of Proposition 3.7 for $p = 1$, but with $\ln n$ replaced everywhere by 1 because we know that $c_1^n = l_2^n \otimes_\pi l_2^n$ has $N = \exp(bn)$ a_1-essential extreme points by Lemma 3.5. ∎

PROPOSITION 3.9. *Let* $1 < p \leqslant 2$. *Then with constants independent of* $1 \leqslant k \leqslant n$, *but depending on* p,

$$a_k(Id\colon l_p^n \otimes_\pi l_p^n \to l_{p'}^n \otimes_\varepsilon l_{p'}^n) \sim \begin{cases} 1 & 1 \leqslant k \leqslant [n^{2/p'}] \\ n^{1/p'}/\sqrt{k} & [n^{2/p'}] < k \leqslant n^2/2 \\ n^{-2/p} & n^2 - [n^{2/p'}] \leqslant k < n^2. \end{cases}$$

For $n^2/2 < k < n^2 - [n^{2/p'}]$ one has

$$a_k(Id\colon l_p^n \otimes_\pi l_p^n \to l_{p'}^n \otimes_\varepsilon l_{p'}^n) \leqslant d_p \sqrt{n^2-k}/n^{1+1/p'}.$$

Proof. The upper estimate for $k \leqslant n^2/2$ again follows from Theorem 2.1 and

$$\mathbb{E}\left\| \sum_{i,j=1}^{n} g_{i,j} e_i \otimes e_j \right\|_{l_{p'}^n \otimes_\varepsilon l_{p'}^n} \leqslant d_p n^{1/p'},$$

which holds by Chevet's inequality (2.1). The upper estimate for $k > n^2/2$ again follows from Theorem 3.4 using Lemma 3.5. For the lower estimate in the case $k \leqslant n^2/2$ we again use Proposition 2.3. To do so, one has to estimate

$$\|Id\colon l_{p'}^n \otimes_\varepsilon l_{p'}^n \to l_2^{n^2}\|$$

from above by $n^{1/p}$. This is seen as follows: For any $T \in L(\mathbb{R}^n, \mathbb{R}^n)$,

$$\begin{aligned}\|T\|_{l_2^{n^2}} = \pi_2(T\colon l_2^n \to l_2^n) &\leqslant \|Id\colon l_2^n \to l_p^n\| \; \|T\colon l_p^n \to l_{p'}^n\| \\ &\quad \times \pi_2(Id\colon l_{p'}^n \to l_2^n) \\ &\leqslant n^{-(1/2)+1/p} \|T\|_{l_{p'}^n \otimes_\varepsilon l_{p'}^n} \times n^{1/2} = n^{1/p} \|T\|_{l_{p'}^n \otimes_\varepsilon l_{p'}^n}. \quad \blacksquare\end{aligned}$$

Remark. For $n^2/2 < k < n^2 - [n^{2/p'}]$, the given upper estimate is probably asymptotically optimal. This would follow from Proposition 2.4 if the largest dimension of a $\leqslant 2$ Hilbertian subspace of $l_{p'}^n \otimes_\varepsilon l_{p'}^n$ could be shown to be of order $n^{2/p'}$ (as in the case of $l_{p'}^n$, cf. [3]), which is true in the case of $p = 2$ at least.

Note added in proof. After finishing this paper, the following result concerning the duality problem for entropy numbers was proved by H. König and V. Milman (On the covering number of convex bodies, to appear): For any $\lambda > 0$ there is $d = d(\lambda) > 1$ such that for any finite rank operator $v\colon X \to Y$ between Banach spaces and all $j > \lambda$. rank(v) one has

$$e_{[dj]}(v) \leqslant 2e_j(v^*), \qquad e_{[dj]}(v^*) \leqslant 2e_j(v).$$

Moreover, $d(\lambda) \to 1$ for $\lambda \to \infty$.

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